

STABILITY OF SOLITARY-WAVE SOLUTIONS OF COUPLED NLS EQUATIONS WITH POWER-TYPE NONLINEARITIES

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ABSTRACT. This paper proves existence and stability results of solitary-wave solutions of a system of 2-coupled nonlinear Schrödinger equations with power-type nonlinearities arising in several models of modern physics. The existence of vector solitary-wave solutions (i.e, both components are nonzero) is established via variational methods. The set of minimizers is shown to be stable and further information about the structures of this set are given. The results extend stability results previously obtained by Cipolatti and Zumpichiatti [14], Nguyen and Wang [31, 32], and Ohta [33].

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

$$(1.1) \quad iu_t + u_{xx} + |u|^{p-1}u = 0,$$

where u is a complex-valued function of $(x, t) \in \mathbb{R}^2$, has been widely recognized as a universal mathematical model for describing the evolution of a slowly varying wave packet in a general nonlinear wave system. It plays an important role in a wide range of physical subjects such as plasma physics [21], nonlinear optics [1], hydrodynamics [42], magnetic systems [19], to name a few. The NLS equation has been also derived as the modulation equation for wave packets in spatially periodic media such as photonic band gap materials and Bose-Einstein condensates [15, 17].

In certain physical situations, when there are two wavetrains moving with nearly the same group velocities, their interactions are then governed by the coupled NLS equations [34, 39]. For example, the coupled NLS systems appear in the study of interactions of waves with different polarizations [8], the description of nonlinear modulations of two monochromatic waves [30], the interaction of Bloch-wave packets

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in a periodic system [35], the evolution of two orthogonal pulse envelopes in birefringent optical fiber [29], the evolution of two surface wave packets in deep water [34], to name a few. The motivation for studying the coupled NLS systems also come from their applications in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states [22].

In this paper we consider the following system of coupled 1-dimensional time-dependent nonlinear Schrödinger equations:

$$(1.2) \quad \begin{cases} iu_t + u_{xx} + (\alpha|u|^{p-2} + \tau|v|^q|u|^{q-2})u = 0 \\ iv_t + v_{xx} + (\beta|v|^{r-2} + \tau|u|^q|v|^{q-2})v = 0, \end{cases}$$

where u, v are complex-valued functions of the real variables x and t , and the constants α, β, τ are real.

The energy H and the component mass Q for the system (1.2) are defined, respectively, as

$$(1.3) \quad H(u, v) = |u_x|_2^2 + |v_x|_2^2 - (a|u|_p^p + b|v|_r^r + c|uv|_q^q),$$

$$(1.4) \quad Q(u) = |u|_2^2,$$

and

$$(1.5) \quad Q(v) = |v|_2^2,$$

where $a = 2\alpha/p$, $b = 2\beta/r$, and $c = 2\tau/q$. The conservation of these functionals is an important ingredient in our stability analysis. (Here $|\cdot|_p$ denote the L^p norm of complex-valued measurable functions on the line. For more details on our notation, see below.)

Solitary-wave solutions of (1.2) are, by definition, solutions of the form

$$(1.6) \quad \begin{aligned} u(x, t) &= e^{i(\omega_1 - \sigma^2)t + i\sigma x + i\lambda_1} \Phi(x - 2\sigma t), \\ v(x, t) &= e^{i(\omega_2 - \sigma^2)t + i\sigma x + i\lambda_2} \Psi(x - 2\sigma t), \end{aligned}$$

where $\omega_1, \omega_2, \sigma \in \mathbb{R}$, and $\Phi, \Psi : \mathbb{R} \rightarrow \mathbb{C}$ are functions of one variable whose values are small when $|\xi| = |x - 2\sigma t|$ is large. Notice that if we insert (1.6) into (1.2), we see that (Φ, Ψ) solves the following system of ordinary differential equations

$$(1.7) \quad \begin{cases} -\Phi'' + \omega_1 \Phi = \alpha|\Phi|^{p-2}\Phi + \tau|\Psi|^q|\Phi|^{q-2}\Phi, \\ -\Psi'' + \omega_2 \Psi = \beta|\Psi|^{r-2}\Psi + \tau|\Phi|^q|\Psi|^{q-2}\Psi. \end{cases}$$

The special case of (1.6) when $\sigma = \lambda_1 = \lambda_2 = 0$, solutions of the form

$$(1.8) \quad (u(x, t), v(x, t)) = (e^{i\omega_1 t} \Phi_{\omega_1}(x), e^{i\omega_2 t} \Psi_{\omega_2}(x)),$$

are usually referred as standing-wave solutions. It is easy to see that (u, v) as defined in (1.8) is a solution of (1.2) if and only if $(\Phi_{\omega_1}, \Psi_{\omega_2})$ is a critical point for the functional $H(u, v)$, when u and v are varied subject to the constraints that $Q(u)$ and $Q(v)$ be held constant. If $(\Phi_{\omega_1}, \Psi_{\omega_2})$ is not only a critical point, but in fact a global minimizer of the constrained variational problem for $H(u, v)$, then (1.8) is called a ground-state solution of (1.2). In some cases, namely when $p = r = 2q = 4$ and under certain conditions on α, β , and τ , it is possible to show further that the ground-state solutions are solitary waves with the usual sech-profile (see, for example, [33, 31]).

Over the past ten years, the existence of nontrivial solutions of the elliptic system (1.7) has been investigated by many authors using different methods. In the case of a positive coupling parameter τ , Maia et al. [27] studied the existence result for positive solutions of (1.7) using constrained minimization methods. They proved the existence of vector ground states of (1.7) i.e., minimal action solutions (Φ, Ψ) with both Φ, Ψ nontrivial. Moreover, the authors gave sufficient conditions for ground states to be positive in both components which basically require the coupling parameter τ to be positive and sufficiently large. Also, Ambrosetti and Colorado [5] and de Figueiredo and Lopes [16] have proved the additional sufficient conditions for the existence of positive ground-state solutions in the special case $p = r = 2q = 4$. Furthermore, for $p = r = 2q = 4$ and small positive values of τ , Lin and Wei [25] and Sirakov [36] proved the existence of positive solutions which have minimal energy among all fully nontrivial solutions. In the repulsive case $\tau < 0$, Mandel [28] recently established existence and nonexistence results concerning fully nontrivial minimal energy solutions. In all these papers, the analysis of their constrained minimization problems does not establish the stability property of solutions. In order to study the stability questions, one has to tackle a different variational formulation.

Our aim here is to prove the stability of vector solitary-wave solutions of the coupled nonlinear Schrödinger system (1.2). The extensive mathematical literature on the subject of stability of solitary waves began with the work Benjamin [6] (see also Bona [10]) for the KdV equation. In subsequent works, many techniques have been developed to refine and extend Benjamin's original conception in many ways to include numerous equations and systems such as Benjamin-Ono equation, intermediate long wave equation, nonlinear Schrödinger equation, Boussinesq systems, etc. For instance, Cazenave and Lions [13] developed a method to prove existence and stability of solitary waves

when they are minimizers of the energy functional and when a compactness condition on minimizing sequences holds. Using the concentration compactness principle of Lions [26], they proved that the solution of (1.1) of the form $e^{i\omega t}\Phi(x)$, $\omega > 0$, and $\Phi(x)$ real and positive, is stable if $p < 5$ in the following sense, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \Phi\|_{H^1(\mathbb{R})} < \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists for all t and

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \inf_{y \in \mathbb{R}} \|u(t) - e^{i\theta}\Phi(\cdot - y)\|_{H^1(\mathbb{R})} < \epsilon.$$

On the other hand, it was shown that solution of the form $e^{i\omega t}\Phi(x)$ for the equation (1.1) is unstable for any $\omega > 0$ if $p \geq 5$ (see Berestycki and Cazenave [7] for $p > 5$, and Weinstein [41] for $p = 5$). The Cazenave and Lions method has since been adapted by different authors to prove existence and stability results of a variety of nonlinear dispersive equations (see, for example, [2, 3, 9, 14, 31, 33]).

We now present a brief discussion of what is currently known about the stability of solitary-wave solutions for (1.2). In the special case $p = r = 2q = 4$, $\alpha = \beta > -1$, and $\tau = 1$, (also known as the symmetric case), the coupled nonlinear Schrödinger system (1.2) is known to have explicit solitary-wave solutions of the form (see, for example, [30, 40])

$$(u_\Omega, v_\Omega) = (e^{i(\Omega - \sigma^2)t + i\sigma x + i\lambda_1} \phi_\Omega(x - 2\sigma t), e^{i(\Omega - \sigma^2)t + i\sigma x + i\lambda_2} \phi_\Omega(x - 2\sigma t)),$$

where $\Omega > 0$, σ , λ_1 , and λ_2 are real constants, and

$$(1.10) \quad \phi_\Omega(x) = \sqrt{\frac{2\Omega}{\alpha + 1}} \text{sech}(\sqrt{\Omega}x).$$

This solution describes a 2-component solitary-wave solutions with the components of equal amplitude. It corresponds to a straight line $\omega_1 = \omega_2$ in the parameter plane (ω_1, ω_2) of a general two-parameter family of solitary waves of (1.2). For this particular form of solitary wave, stability was proved by Ohta [33]. In [31], the stability result in [33] was extended to include a more general setting. Namely, when $p = r = 2q = 4$, and $0 < \tau < \min\{\alpha, \beta\}$; or $\tau > \max\{\alpha, \beta\}$ and $\tau^2 > \alpha\beta$, they proved the stability of solitary-wave solutions of the form

$$\begin{aligned} u_\Omega(x, t) &= e^{i(\Omega - \sigma^2)t + i\sigma x + i\lambda_1} \sqrt{\frac{\tau - \beta}{\tau^2 - \alpha\beta}} \phi_\Omega(x - 2\sigma t), \\ v_\Omega(x, t) &= e^{i(\Omega - \sigma^2)t + i\sigma x + i\lambda_2} \sqrt{\frac{\tau - \alpha}{\tau^2 - \alpha\beta}} \phi_\Omega(x - 2\sigma t), \end{aligned}$$

where $\Omega > 0$, σ , λ_1 , and λ_2 are real constants, and ϕ_Ω as defined in (1.10). In [14] and [16], the stability results were proved by considering

different variational settings than the one used in [31]. For example, in [14], the authors considered the variational problem of finding minimizers of H subject to one constraint being the sum of L^2 -norms of the two components. This variational problem can have different solitary-wave solutions. In fact, the last two pages of [14] show that in the case when

$$\alpha = \beta = \sqrt{\frac{\tau - \beta}{\tau^2 - \alpha\beta}} \text{ and } \tau < \sqrt{\frac{\tau - \beta}{\tau^2 - \alpha\beta}},$$

the solitary-waves which solve the variational problem in [31] are not the same as the solitary waves which solve the variational problem in [14].

In all papers mentioned in the preceding paragraph, the stability results were proved by using variational methods in which constraint functionals were not independently chosen. It is not clear whether the sets of solitary waves obtained from these papers constitute a true two-parameter family of disjoint sets. To obtain a true two-parameter family of solitary waves, one has to characterize solitary waves as minimizers of the energy functional subject to two independent constraints. In [3], the authors proved existence of a true two-parameter family of solitary waves in the context of NLS-KdV system, improving the existence result obtained previously in [20]. Their method also lead to the stability property of solitary waves. Recently, following the same arguments used in [3], Nguyen and Wang [32] proved the stability of a two-parameter family of solitary waves for the NLS system (1.2) in the special case $p = r = 2q = 4$. Here we are able to prove existence and stability of solitary-wave solutions of (1.2) for all $\alpha, \beta, \tau > 0$, and for the range $2 < p, r, 2q < 6$. We will follow the arguments used in [3] to solve a constrained minimization problem. This approach allows us to obtain existence and stability results concerning a true two-parameter family of solitary waves with both component positive, i.e., each component is of the form $e^{i\theta t}p(x)$ with $\theta \in \mathbb{R}$ and $p(x)$ a real-valued positive function in $H^1(\mathbb{R})$.

Logically, prior to a discussion of stability in terms of perturbations of the initial data should be a theory for the initial-valued problem itself. This issue has been studied in [18] (see also [12]). It is proved in [18] that for the range $2 < p, r, 2q < 6$, for any $(u(x, 0), v(x, 0)) \in Y$, there exists a unique solution $(u(x, t), v(x, t))$ of one-dimensional coupled NLS system (1.2) in $C(\mathbb{R}, Y)$ emanating from $(u(x, 0), v(x, 0))$, and $(u(x, t), v(x, t))$ satisfies

$$Q(u(x, t)) = Q(u(x, 0)), \quad Q(v(x, t)) = Q(v(x, 0)),$$

and

$$H(u(x, t), v(x, t)) = H(u(x, 0), v(x, 0)).$$

However there are some restriction on the applicable range of p, r, q in higher dimension (See [18, 12] for more details).

We now describe briefly our results. The existence of solitary waves is obtained by studying constrained minimization problem and applying the concentration-compactness lemma of P. L. Lions [26]. More precisely, for $s > 0$ and $t > 0$, we define

$$(1.11) \quad \Sigma_{s,t} = \{(f, g) \in Y : Q(f) = s, Q(g) = t\}.$$

and consider the problem of finding minimizers of the functional $H(f, g)$ subject to $(f, g) \in \Sigma_{s,t}$. To prevent dichotomy of minimizing sequences while applying concentration-compactness method, one require to prove the strict subadditivity of the variational problem with respect to the constraint parameters. More precisely, we require to prove strict subadditivity of the function

$$(1.12) \quad \Theta(s, t) = \inf \{H(f, g) : (f, g) \in \Sigma_{s,t}\}.$$

We establish the strict subadditivity of $\Theta(s, t)$ following the ideas and results contained in [3], which utilize the fact that the H^1 -norms of some functions are strictly decreasing when the mass of the functions are symmetrically rearranged. The set of minimizers, namely

$$(1.13) \quad \mathcal{F}_{s,t} = \{(\Phi, \Psi) \in Y : H(\Phi, \Psi) = \Theta(s, t), (\Phi, \Psi) \in \Sigma_{s,t}\}.$$

is shown to be stable in the sense that a solution which starts near the set will remain near it for all time. We also consider the question about the characterization of the set $\mathcal{F}_{s,t}$.

The following are our existence and stability results.

Theorem 1.1. *Suppose $\alpha, \beta, \tau > 0$ and $2 < p, r, 2q < 6$.*

- (a) *The function $\Theta(s, t)$ defined in (1.12) is finite, and if $\{(f_n, g_n)\}$ is any sequence in Y such that*

$$(1.14) \quad \lim_{n \rightarrow \infty} Q(f_n) = s, \quad \lim_{n \rightarrow \infty} Q(g_n) = t, \quad \text{and} \quad \lim_{n \rightarrow \infty} H(f_n, g_n) = \Theta(s, t),$$

then there exists a subsequence $\{(f_{n_k}, g_{n_k})\}$ and a family $\{y_k\} \subset \mathbb{R}$ such that $\{(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))\}$ converges strongly in Y to some (Φ, Ψ) in $\mathcal{F}_{s,t}$. In particular, the set $\mathcal{F}_{s,t}$ is non-empty.

- (b) *Each pair $(\Phi, \Psi) \in \mathcal{F}_{s,t}$ is a solution of (1.7) for some $\omega_1 > 0$ and $\omega_2 > 0$, and thus when inserted into (1.6) yields a two-parameter solitary-wave solution to the system (1.2).*

- (c) For each pair (Φ, Ψ) in $\mathcal{F}_{s,t}$, there exist numbers $\theta_1, \theta_2 \in \mathbb{R}$ and functions $\tilde{\phi}$ and $\tilde{\psi}$ such that $\tilde{\phi}(x), \tilde{\psi}(x) > 0$ for all $x \in \mathbb{R}$, and

$$\Phi(x) = e^{i\theta_1} \tilde{\phi}(x) \text{ and } \Psi(x) = e^{i\theta_2} \tilde{\psi}(x).$$

Moreover, the functions Φ and Ψ are infinitely differentiable on \mathbb{R} .

- (d) For any $s, t > 0$, the set $\mathcal{F}_{s,t}$ is stable in the following sense: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(u_0, v_0) \in Y$ satisfies

$$\inf\{\|(u_0, v_0) - (\Phi, \Psi)\|_Y : (\Phi, \Psi) \in \mathcal{F}_{s,t}\} < \delta,$$

and $(u(x, t), v(x, t))$ is the solution of (1.2) with $(u(x, 0), v(x, 0)) = (u_0, v_0)$, then for all $t \geq 0$,

$$\inf\{\|(u(\cdot, t), v(\cdot, t)) - (\Phi, \Psi)\|_Y : (\Phi, \Psi) \in \mathcal{F}_{s,t}\} < \epsilon.$$

The method presented in this paper should be easily extendable to versions of (1.2) with combined power-type nonlinearities, such as the following system of coupled nonlinear Schrödinger equations

$$(1.15) \quad \begin{cases} iu_t + u_{xx} + \alpha|u|^{p-2}u + \sum_{k=1}^m \tau|v|^{q_k}|u|^{q_k-2}u = 0 \\ iv_t + v_{xx} + \beta|v|^{r-2}v + \sum_{k=1}^m \tau|u|^{q_k}|v|^{q_k-2}v = 0. \end{cases}$$

The global existence of the solutions of this system is studied in [37]. The energy functional K defined by

$$K(u, v) = \frac{1}{2} (|u_x|_2^2 + |v_x|_2^2) - \frac{1}{p} \left(\alpha|u|_p^p + \beta|v|_r^r + \tau \sum_{k=1}^m |uv|_{q_k}^{q_k} \right)$$

is conserved for the flow defined by (1.15). The functionals $Q(u)$ and $Q(v)$ defined above are conserved functionals for (1.15) as well. Our method can be applied to prove an analogue of Theorem 1.1 concerning existence and stability results of vector solitary-wave solutions to (1.15) for all $\alpha, \beta, \tau > 0$, and all $2 < p, r, 2q_k < 6$ ($k = 1, 2, \dots, m$).

Notation. For $1 \leq s \leq \infty$, the space of complex measurable functions whose s -th power is integrable will be denoted by $L^s = L^s(\mathbb{R})$ and its standard norm by $|f|_s$,

$$|f|_s = \left(\int_{-\infty}^{\infty} |f|^s dx \right)^{1/s} \quad \text{for } 1 \leq s < \infty$$

and $|f|_\infty$ is the essential supremum of $|f|$ on \mathbb{R} . We denote by $H^1(\mathbb{R})$ the Sobolev space of all complex-valued, measurable functions defined

on \mathbb{R} such that both f and f' are in L^2 . The norm $\|\cdot\|_1$ on H^1 is defined by

$$\|f\|_1 = \left(\int_{-\infty}^{\infty} (|f|^2 + |f'|^2) \right)^{1/2}.$$

In particular, we use $\|f\|$ to denote the L^2 norm of a function f . We define the space Y to be the Cartesian product $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, furnished with the norm

$$\|(f, g)\|_Y^2 = \|f\|_1^2 + \|g\|_1^2.$$

The letter C will denote various positive constants whose exact values may change from line to line but are not essential in the course of the analysis.

2. EXISTENCE AND STABILITY RESULTS

We assume throughout this paper, unless otherwise stated, that the assumptions $\alpha, \beta, \tau > 0$, and $2 < p, r, 2q < 6$ hold.

To each minimizing sequence $\{(f_n, g_n)\}$ of $\Theta(s, t)$, we associate a sequence of nondecreasing functions $M_n : [0, \infty) \rightarrow [0, s + t]$ defined by

$$M_n(\zeta) = \sup_{y \in \mathbb{R}} \int_{y-\zeta}^{y+\zeta} \rho_n(x) \, dx.$$

where $\rho_n(x) := |f_n(x)|^2 + |g_n(x)|^2$. An elementary argument (by Helly's selection theorem, for example) shows that any uniformly bounded sequence of nondecreasing functions on $[0, \infty)$ must have a subsequence which converges pointwise (in fact, uniformly on compact sets) to a nondecreasing limit function on $[0, \infty)$. Thus, $M_n(\zeta)$ has such a subsequence (see Lemma 2.1 below), which we again denote by M_n . Let $M(\zeta) : [0, \infty) \rightarrow [0, s + t]$ be the nondecreasing function to which M_n converges, and define

$$(2.1) \quad \gamma = \lim_{\zeta \rightarrow \infty} M(\zeta).$$

Then γ satisfies $0 \leq \gamma \leq s + t$. From Lions' Concentration Compactness Lemma (see [26]), there are three possibilities for the value of γ that correspond to three distinct types of limiting behavior of the sequence $\rho_n(x)$ as $n \rightarrow \infty$, which are suggestively labeled by Lions as 'vanishing', 'dichotomy' and 'compactness', respectively:

- (a) Case 1 : (*Vanishing*) $\gamma = 0$. Since $M(\zeta)$ is non-negative and nondecreasing, this is equivalent to saying

$$M(\zeta) = \lim_{n \rightarrow \infty} M_n(\zeta) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-\zeta}^{y+\zeta} \rho_n(x) \, dx = 0,$$

for all $\zeta < \infty$, or

- (b) Case 2 : (*Dichotomy*) $\gamma \in (0, s+t)$, or
(c) Case 3 : (*Compactness*) $\gamma = s+t$, that is, there exists $\{y_n\} \subset \mathbb{R}$ such that $\rho_n(\cdot + y_n)$ is tight, namely, for all $\varepsilon > 0$, there exists $\zeta < \infty$ such that for all $n \in \mathbb{N}$,

$$\int_{y_n-\zeta}^{y_n+\zeta} \rho_n(x) dx \geq (s+t) - \varepsilon.$$

The method of concentration compactness, as applied to this situation, consists of the observation that if $\gamma = s+t$, then the minimizing sequence $\{(f_n, g_n)\}$ has a subsequence which, up to translations in the underlying spatial domain, converges strongly in Y to an element of $\mathcal{F}_{s,t}$. Typically, one proves $\gamma = s+t$ by ruling out the other two possibilities. We now give the details of the method, and prove our existence and stability results.

We first establish some properties of $\Theta(s, t)$ and its minimizing sequences which are independent of the value γ .

Lemma 2.1. *If $\{(f_n, g_n)\}$ is a minimizing sequence $\Theta(s, t)$, then there exists constants $B > 0$ such that*

$$\|f_n\|_1 + \|g_n\|_1 \leq B \text{ for all } n.$$

Moreover, for every $s, t > 0$, one has $-\infty < \Theta(s, t) < 0$.

Proof. Using the Gagliardo-Nirenberg inequality, we have

$$(2.2) \quad \|f_n\|_p^p \leq C \|f_{nx}\|^{(p-2)/2} \cdot \|f_n\|^{(p+2)/2}.$$

Since $\{(f_n, g_n)\}$ is a minimizing sequence, both $\|f_n\|$ and $\|g_n\|$ are bounded. Then, from (2.2), we obtain

$$(2.3) \quad \|f_n\|_p^p \leq C \|f_{nx}\|^{(p-2)/2} \leq C \|(f_n, g_n)\|_Y^{(p-2)/2},$$

where C denotes various constants which are independent of f_n and g_n . Similarly, we have the following estimate

$$(2.4) \quad \|g_n\|_r^r \leq C \|g_{nx}\|^{(r-2)/2} \leq C \|(f_n, g_n)\|_Y^{(r-2)/2}.$$

From Cauchy-Schwartz inequality, we also have

$$(2.5) \quad |f_n g_n|_q^q dx \leq \frac{1}{2} (|f_n|_{2q}^{2q} + |g_n|_{2q}^{2q}) \leq C \|(f_n, g_n)\|_Y^{q-1}.$$

Now, we write

$$\begin{aligned} \|(f_n, g_n)\|_Y^2 &= \|f_n\|_1^2 + \|g_n\|_1^2 \\ &= H(f_n, g_n) + (a|f_n|_p^p + b|g_n|_r^r + c|f_n g_n|_q^q) dx + (s+t). \end{aligned}$$

Since $H(f_n, g_n)$ is bounded, we obtain

$$\|(f_n, g_n)\|_Y^2 \leq C \left(1 + \|(f_n, g_n)\|_Y^{(p-2)/2} + \|(f_n, g_n)\|_Y^{(r-2)/2} + \|(f_n, g_n)\|_Y^{q-1} \right),$$

As the norm of the minimizing sequence $\{(f_n, g_n)\}$ is bounded by itself but with smaller power, the existence of the desired bound B follows.

To see $\Theta(s, t) > -\infty$, it suffices to bound $H(f, g)$ from below by a number which is independent of f and g . Using the estimates (2.3), (2.4), and (2.5), we obtain for $(f, g) \in \Sigma_{s,t}$,

$$\begin{aligned} H(f, g) &\geq \|f_x\|^2 + \|g_x\|^2 - C\|f_x\|^{(p-2)/2} - C\|g_x\|^{(r-2)/2} \\ &\quad - C(\|f_x\|^{q-1} + \|g_x\|^{q-1}), \end{aligned}$$

where C denotes various constants independent of f and g . Let us define

$$Z(x, y) = |x|^2 + |y|^2 - C(|x|^{(p-2)/2} + |y|^{(r-2)/2} + |x|^{q-1} + |y|^{q-1}).$$

Since $2 < p, r, 2q < 6$, we have $\varrho := \min Z(x, y) > -\infty$. In particular, for all $(f, g) \in \Sigma_{s,t}$, we have that

$$H(f, g) \geq Z(\|f_x\|, \|g_x\|) \geq \varrho > -\infty.$$

To see that $\Theta(s, t) < 0$, choose $(f, g) \in \Sigma_{s,t}$, and $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. For each $\theta > 0$, the functions $f_\theta(x) = \theta^{1/2}f(\theta x)$ and $g_\theta(x) = \theta^{1/2}g(\theta x)$ satisfy $(f_\theta, g_\theta) \in \Sigma_{s,t}$, and

$$\begin{aligned} H(f_\theta, g_\theta) &= \int_{-\infty}^{\infty} (|f_{\theta x}|^2 + |g_{\theta x}|^2 - a|f_\theta|^p - b|g_\theta|^r - c|f_\theta|^q|g_\theta|^q) dx \\ &\leq \theta^2 \int_{-\infty}^{\infty} (|f_x|^2 + |g_x|^2) dx - \theta^{q-1} \int_{-\infty}^{\infty} c|f|^q|g|^q dx. \end{aligned}$$

Hence, by taking θ sufficiently small, we get $H(f_\theta, g_\theta) < 0$. \square

Lemma 2.2. *Let (f_n, g_n) be a minimizing sequence for $\Theta(s, t)$. Then for all sufficiently large n ,*

- (i) *if $t > 0$ and $s \geq 0$, then $\exists \delta_1 > 0$ such that $\|g_{nx}\| \geq \delta_1$.*
- (ii) *if $s > 0$ and $t \geq 0$, then $\exists \delta_2 > 0$ such that $\|f_{nx}\| \geq \delta_2$.*

Proof. Suppose to the contrary that (i) is false. Then, by passing to a subsequence if necessary, we may assume there exists a minimizing sequence for which $\lim_{n \rightarrow \infty} \|g_{nx}\| = 0$. By Gagliardo-Nirenberg inequalities, it then follows that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |g_n|^r dx = 0 \text{ and } \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_n|^q |g_n|^q dx = 0.$$

Therefore, we have that

$$\begin{aligned} \Theta(s, t) &= \lim_{n \rightarrow \infty} H(f_n, g_n) \\ (2.6) \quad &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|f_{nx}|^2 - a|f_n|^p) dx. \end{aligned}$$

Pick any non-negative function ψ such that $\|\psi\|^2 = t$. For every $\theta > 0$, the function $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$ satisfies $\|\psi_\theta\|^2 = t$, and hence, for all n ,

$$\Theta(s, t) \leq H(f_n, \psi_\theta).$$

On the other hand, if we define

$$(2.7) \quad \eta = \theta^2 \int_{-\infty}^{\infty} |\psi_x|^2 dx - \theta^{(r-2)/2} \int_{-\infty}^{\infty} b|\psi|^r dx,$$

then $\eta < 0$ for sufficiently small θ . Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} \Theta(s, t) &\leq H(f_n, \psi_\theta) \\ &\leq \int_{-\infty}^{\infty} (|f_{nx}|^2 - a|f_n|^p) dx + \eta. \end{aligned}$$

Consequently

$$\Theta(s, t) \leq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|f_{nx}|^2 - a|f_n|^p) dx + \eta,$$

which contradicts (2.6) and (2.7). The case (ii) can be proved similarly. \square

Lemma 2.3. *Let $1 < \alpha < 5$ and $\beta > 0$. Define $J : H^1(\mathbb{R}) \rightarrow \mathbb{R}$ by*

$$(2.8) \quad J(h(x, t)) = \int_{-\infty}^{\infty} (|h_x(x, t)|^2 - \beta|h(x, t)|^{\alpha+1}) dx.$$

Let $s > 0$, and let $\{h_n\}$ be any sequence in H^1 such that $\|h_n\|^2 \rightarrow s$ and

$$\lim_{n \rightarrow \infty} J(h_n) = \inf \{J(h) : h \in H^1 \text{ and } \|h\|^2 = s\}.$$

Then there exists a subsequence $\{h_{n_k}\}$, a family $\{y_k\} \subset \mathbb{R}$, and a real number θ such that $e^{-i\theta}h_{n_k}(x + y_k)$ converges strongly in H^1 norm to $h_s(x)$, where

$$(2.9) \quad h_s(x) = \left(\frac{\lambda}{\beta}\right)^{1/(\alpha-1)} \operatorname{sech}^{2/(\alpha-1)} \left(\frac{\sqrt{\lambda}(\alpha-1)x}{2}\right),$$

and $\lambda > 0$ is chosen so that $\|h_s\|^2 = s$. In particular,

$$(2.10) \quad J(h_s) = \inf \{J(h) : h \in H^1 \text{ and } \|h\|^2 = s\}.$$

Proof. The fact that some translated subsequence of h_n must converge strongly in H^1 norm can be proved by the use of Cazenave-Lions method (see, for example, [13, 12]).

Let $\varphi \in H^1$ be the limit of the translated subsequence $\{h_{n_k}(x + \tilde{y}_k)\}$ of $\{h_n\}$. Then the limit function φ satisfies

$$(2.11) \quad J(\varphi) = \inf \{J(h) : h \in H^1 \text{ and } \|h\|^2 = s\},$$

and also be a solution of

$$(2.12) \quad -2\varphi'' - (\alpha + 1)\beta\varphi^\alpha = -2\lambda\varphi$$

for some real number λ . It is well known (see Theorem 8.1.6 of [12]) that the solutions of (2.12) can be described explicitly by

$$\{e^{i\theta}h_s(\cdot + y_s), y_s, \theta \in \mathbb{R}\}.$$

Then (2.10) follows from (2.11). Also, if we define $y_k = \tilde{y}_k - y_s$, then we have that $e^{-i\theta}h_{n_k}(x + y_k)$ converges in H^1 to h_s . \square

Lemma 2.4. *Suppose (f_n, g_n) is a minimizing sequence for $\Theta(s, t)$, where $s > 0$ and $t > 0$. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for all sufficiently large n ,*

$$|f_{nx}|_2^2 - a|f_n|_p^p - c|f_n g_n|_q^q \leq -\delta_1, \text{ and } |g_{nx}|_2^2 - b|g_n|_r^r - c|f_n g_n|_q^q \leq -\delta_2.$$

Proof. Both inequalities can be proved by using similar arguments. We only prove the first inequality. Suppose the conclusion is false. Then, by passing to a subsequence if necessary, we may assume that there exists a minimizing sequence (f_n, g_n) for which

$$(2.13) \quad \liminf_{n \rightarrow \infty} (|f_{nx}|_2^2 - a|f_n|_p^p - c|f_n g_n|_q^q) \geq 0,$$

and so

$$(2.14) \quad \Theta(s, t) = \lim_{n \rightarrow \infty} H(f_n, g_n) \geq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|g_{nx}|^2 - b|g_n|^r) dx.$$

Define J and g_t as in Lemma 2.3 with $h = g$, $\beta = b$, and $\alpha = r - 1$. Then (2.14) implies that

$$(2.15) \quad \Theta(s, t) \geq J(g_t).$$

On the other hand, take any $f \in H^1$ such that $\|f\|^2 = s$ and

$$(2.16) \quad |f_x|_2^2 - a|f|_p^p - c|f g_t|_q^q < 0.$$

To construct such a function f , take an arbitrary smooth, non-negative function ψ with compact support such that $\psi(0) = 1$ and $\|\psi\| = s$, and for $\theta > 0$, define $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$. Then, $f = \psi_\theta$ satisfies (2.16) for sufficiently small θ . Therefore,

$$(2.17) \quad \Theta(s, t) \leq H(f, g_t) \leq |f_x|_2^2 - a|f|_p^p - c|f g_t|_q^q + J(g_t) < J(g_t),$$

which contradicts (2.15), and hence lemma follows. \square

Lemma 2.5. $H(|f|, |g|) \leq H(f, g)$ for all $(f, g) \in Y$.

Proof. The proof follows from the fact that if $f \in H^1$, then $|f(x)|$ is in H^1 and

$$(2.18) \quad \int_{-\infty}^{\infty} ||f|_x|^2 \, dx \leq \int_{-\infty}^{\infty} |f_x|^2 \, dx.$$

A proof of (2.18) can be given by working with Fourier transforms of f and $|f|$ and is easily constructed by adapting the proof of Lemma 3.5 in [4]. \square

In the sequel, we denote by $e^*(x)$ the symmetric decreasing rearrangement for a function $e : \mathbb{R} \rightarrow [0, \infty)$. We refer the reader to [24] for details about symmetric decreasing rearrangements. We note here that if $(f, g) \in Y$, then $|f|, |g| \in Y$, and hence symmetric rearrangements $|f|^*$ and $|g|^*$ of $|f|$ and $|g|$ are well-defined. A basic property about symmetric decreasing rearrangement is that L^p norms are preserved:

$$(2.19) \quad \int_{-\infty}^{\infty} (|f|^*)^p \, dx = \int_{-\infty}^{\infty} |f|^p \, dx$$

Lemma 2.6. $H(|f|^*, |g|^*) \leq H(f, g)$ for all $(f, g) \in Y$.

Proof. From Theorem 3.4 of [24], we have

$$(2.20) \quad \int_{-\infty}^{\infty} (|f|^*)^q (|g|^*)^q \, dx \geq \int_{-\infty}^{\infty} |f|^q |g|^q \, dx.$$

Lemma 7.17 of [24] implies that

$$(2.21) \quad \int_{-\infty}^{\infty} |(|f|^*)_x|^2 \, dx \leq \int_{-\infty}^{\infty} ||f|_x|^2 \, dx,$$

and similarly for $g(x)$. Then, the claim follows by using the facts (2.19), (2.20), (2.21), and Lemma 2.5. \square

The next lemma is one-dimensional version of Proposition 1.4 of [11]. A proof of this lemma is given in [3] (see also [23]).

Lemma 2.7. *Suppose $u, v : \mathbb{R} \rightarrow [0, \infty)$ are even, C^∞ , non-increasing, and have compact support in \mathbb{R} . Let a_1 and a_2 be real numbers such that $\text{supp}(u(x + a_1)) \cap \text{supp}(v(x + a_2)) = \emptyset$, and define*

$$e(x) = u(x + a_1) + v(x + a_2).$$

Then the derivative $(e^)'$ of e^* (in sense of distribution) is in L^2 , and satisfies*

$$(2.22) \quad \|(e^*)'\|^2 \leq \|e'\|^2 - \frac{3}{4} \min\{\|u'\|^2, \|v'\|^2\}.$$

The next lemma proves that $\Theta(s, t)$ is subadditive:

Lemma 2.8. *Let $s_1, s_2, t_1, t_2 \geq 0$ be such that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$, and $s_2 + t_2 > 0$. Then*

$$(2.23) \quad \Theta(s_1 + s_2, t_1 + t_2) < \Theta(s_1, t_1) + \Theta(s_2, t_2).$$

Proof. Let $i = 1, 2$. Then, following closely the arguments used in [3], we can choose minimizing sequences $(f_n^{(i)}, g_n^{(i)})$ for $\Theta(s_i, t_i)$ such that $f_n^{(i)}$ and $g_n^{(i)}$ are real-valued, non-negative, even, C^∞ with compact support in \mathbb{R} , non-increasing on $\{x : x \geq 0\}$, and satisfy $(f_n^{(i)}, g_n^{(i)}) \in \Sigma_{s_i, t_i}$.

Now, for each n , choose a number x_n such that $f_n^{(1)}(x)$ and $\tilde{f}_n^{(2)}(x) = f_n^{(2)}(x + x_n)$ have disjoint support, and $g_n^{(1)}(x)$ and $\tilde{g}_n^{(2)}(x) = g_n^{(2)}(x + x_n)$ have disjoint support. Define

$$f_n = \left(f_n^{(1)} + \tilde{f}_n^{(2)}\right)^* \quad \text{and} \quad g_n = \left(g_n^{(1)} + \tilde{g}_n^{(2)}\right)^*.$$

Then $(f_n, g_n) \in \Sigma_{s_1+s_2, t_1+t_2}$, and hence,

$$(2.24) \quad \Theta(s_1 + s_2, t_1 + t_2) \leq H(f_n, g_n).$$

On the other hand, from Lemma 2.7 we have that

$$(2.25) \quad \begin{aligned} \int_{-\infty}^{\infty} (f_{nx}^2 + g_{nx}^2) \, dx &\leq \int_{-\infty}^{\infty} \left((f_n^{(1)} + \tilde{f}_n^{(2)})_x^2 + (g_n^{(1)} + \tilde{g}_n^{(2)})_x^2 \right) \, dx - K_n \\ &= \int_{-\infty}^{\infty} \left((f_{nx}^{(1)})^2 + (\tilde{f}_{nx}^{(2)})^2 + (g_{nx}^{(1)})^2 + (\tilde{g}_{nx}^{(2)})^2 \right) \, dx - K_n, \end{aligned}$$

where

$$(2.26) \quad K_n = \frac{3}{4} \left(\min \{ \|f_{nx}^{(1)}\|^2, \|f_{nx}^{(2)}\|^2 \} + \min \{ \|g_{nx}^{(1)}\|^2, \|g_{nx}^{(2)}\|^2 \} \right).$$

Moreover, from the properties of rearrangements, we have that

$$(2.27) \quad \begin{aligned} \int_{-\infty}^{\infty} |f_n|^p \, dx &= \int_{-\infty}^{\infty} |f_n^{(1)}|^p \, dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^p \, dx, \\ \int_{-\infty}^{\infty} |g_n|^r \, dx &= \int_{-\infty}^{\infty} |g_n^{(1)}|^r \, dx + \int_{-\infty}^{\infty} |g_n^{(2)}|^r \, dx, \\ \int_{-\infty}^{\infty} |f_n|^q |g_n|^q \, dx &\geq \int_{-\infty}^{\infty} |f_n^{(1)}|^q |g_n^{(1)}|^q \, dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^q |g_n^{(2)}|^q \, dx. \end{aligned}$$

Then, (2.24), (2.25) and (2.27) give, for all n ,

$$(2.28) \quad \Theta(s_1 + t_1, s_2 + t_2) \leq H(f_n, g_n) \leq H(f_n^{(1)}, g_n^{(1)}) + H(f_n^{(2)}, g_n^{(2)}) - K_n.$$

Hence, we obtain

$$(2.29) \quad \Theta(s_1 + t_1, s_2 + t_2) \leq \Theta(s_1, t_1) + \Theta(s_2, t_2) - \liminf_{n \rightarrow \infty} K_n.$$

Since $t_1 + t_2 > 0$, we consider the following three cases: (i) $t_1 > 0$ and $t_2 > 0$; (ii) $t_1 = 0$, $t_2 > 0$, and $s_2 > 0$; and (iii) $t_1 = 0$, $t_2 > 0$, and $s_2 = 0$.

Case 1: When $t_1 > 0$ and $t_2 > 0$. Lemma 2.2 guarantees that there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for all sufficiently large n ,

$$\|(g_n^{(1)})_x\| \geq \delta_1 \text{ and } \|(g_n^{(2)})_x\| \geq \delta_2.$$

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, (2.26) gives $K_n \geq 3\delta/4$ for all sufficiently large n . From (2.29) we have

$$\Theta(s_1 + t_1, s_2 + t_2) \leq \Theta(s_1, t_1) + \Theta(s_2, t_2) - 3\delta/4 < \Theta(s_1, t_1) + \Theta(s_2, t_2),$$

as desired.

Case 2: When $t_1 = 0$, $t_2 > 0$, and $s_2 > 0$. Since $s_1 + t_1 > 0$, $s_1 > 0$ too. By Lemma 2.2, there exist numbers $\delta_3 > 0$ and $\delta_4 > 0$ such that for all sufficiently large n ,

$$\|(f_n^{(1)})_x\| \geq \delta_3 \text{ and } \|(f_n^{(2)})_x\| \geq \delta_4.$$

Let $\delta = \min(\delta_3, \delta_4) > 0$. Then, (2.26) gives $K_n \geq 3\delta/4$ for all sufficiently large n . From (2.29) we have

$$\Theta(s_1 + t_1, s_2 + t_2) \leq \Theta(s_1, t_1) + \Theta(s_2, t_2) - 3\delta/4 < \Theta(s_1, t_1) + \Theta(s_2, t_2).$$

Case 3: When $t_1 = 0$, $t_2 > 0$, and $s_2 = 0$. In this case, we have

$$\Theta(0, t_2) = \inf \left\{ \int_{-\infty}^{\infty} (|g_x|^2 - b|g|^r) \, dx : g \in H^1 \text{ and } \|g\|^2 = t_2 > 0 \right\}$$

and

$$\Theta(s_1, 0) = \inf \left\{ \int_{-\infty}^{\infty} (|f_x|^2 - a|f|^p) \, dx : f \in H^1 \text{ and } \|f\|^2 = s_1 > 0 \right\}.$$

Lemma 2.3 with $h = g$, $s = t_2$, $\beta = b$, and $\alpha = r - 1$ implies $\Theta(0, t_2) = J(g_{t_2})$. Similarly, let f_{s_1} be such that $\Theta(s_1, 0) = J(f_{s_1})$. Clearly,

$$\int_{-\infty}^{\infty} |f_{s_1}|^q |g_{t_2}|^q \, dx > 0$$

and so

$$\begin{aligned} \Theta(s_1, t_2) &\leq H(f_{s_1}, g_{t_2}) = \Theta(s_1, 0) + \Theta(0, t_2) - c \int_{-\infty}^{\infty} |f_{s_1}|^q |g_{t_2}|^q \, dx \\ &< \Theta(s_1, 0) + \Theta(0, t_2). \end{aligned}$$

This completes the proof of lemma. \square

Lemma 2.9. *Suppose $\gamma = s + t$ and let $\{(f_n, g_n)\}$ be a minimizing sequence for $\Theta(s, t)$. Then there exists a sequence of real numbers $\{y_n\}$ such that*

1. *for every $z < s + t$ there exists $\zeta = \zeta(z)$ such that*

$$\int_{y_n - \zeta}^{y_n + \zeta} (|f_n|^2 + |g_n|^2) dx > z$$

for all sufficiently large n .

2. *the sequence $\{(w_n, z_n)\}$ defined by*

$$w_n(x) = f_n(x + y_n) \text{ and } z_n(x) = g_n(x + y_n), \quad x \in \mathbb{R},$$

has a subsequence which converges in Y norm to a function $(\Phi, \Psi) \in \mathcal{F}_{s,t}$. In particular, $\mathcal{F}_{s,t}$ is nonempty.

Proof. Since $\gamma = s + t$, then, by the definition of γ , there exists ζ_0 such that for n sufficiently large, $M_n(\zeta_0) > (s + t)/2$. Thus, for each sufficiently large n , we can find y_n such that

$$\int_{y_n - \zeta_0}^{y_n + \zeta_0} (|f_n|^2 + |g_n|^2) dx > \frac{s + t}{2}.$$

Now, let $z < s + t$. Clearly, we may assume $z \in (\frac{s+t}{2}, s + t)$. Again, since $\gamma = s + t$, we can find $\zeta_1 = \zeta_1(z)$, such that for n sufficiently large, $M_n(\zeta_1) > z$, and so, we can choose \tilde{y}_n such that

$$\int_{\tilde{y}_n - \zeta_1}^{\tilde{y}_n + \zeta_1} (|f_n|^2 + |g_n|^2) dx > z$$

for some $\tilde{y}_n \in \mathbb{R}$. Since $\int_{-\infty}^{\infty} (|f_n|^2 + |g_n|^2) dx = s + t$, it follows that for large n , the intervals $[\tilde{y}_n - \zeta_1, \tilde{y}_n + \zeta_1]$ and $[y_n - \zeta_0, y_n + \zeta_0]$ must overlap. Then, by defining $\zeta = 2\zeta_1 + \zeta_0$, we have that $[y_n - \zeta, y_n + \zeta]$ contains $[\tilde{y}_n - \zeta_1, \tilde{y}_n + \zeta_1]$, and the statement 1 follows.

To prove statement 2, notice first that statement 1 implies that, for every $k \in \mathbb{N}$, there exists $\zeta_k \in \mathbb{R}$ such that

$$(2.30) \quad \int_{-\zeta_k}^{\zeta_k} (|w_n|^2 + |z_n|^2) dx > s + t - \frac{1}{k},$$

for all sufficiently large n . Since $\{(w_n, z_n)\}$ is bounded uniformly in Y , there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in Y to a limit $(\Phi, \Psi) \in Y$. Then Fatou's lemma implies that

$$\|\Phi\|^2 + \|\Psi\|^2 \leq \liminf_{n \rightarrow \infty} (\|w_n\|^2 + \|z_n\|^2) = s + t.$$

Moreover, for fixed k , (w_n, z_n) converges weakly in $H^1(-\zeta_k, \zeta_k) \times H^1(-\zeta_k, \zeta_k)$ to (Φ, Ψ) , and therefore has a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges strongly to (Φ, Ψ) in $L^2(-\zeta_k, \zeta_k) \times L^2(-\zeta_k, \zeta_k)$. By a

diagonalization argument, we may assume that the subsequence has this property for every k simultaneously. It then follows from (2.30) that

$$\|\Phi\|^2 + \|\Psi\|^2 \geq \int_{-\zeta_k}^{\zeta_k} (|\Phi|^2 + |\Psi|^2) dx \geq s + t - \frac{1}{k}.$$

Since k was arbitrary, we get

$$\|\Phi\|^2 + \|\Psi\|^2 = s + t,$$

which implies that (w_n, z_n) converges strongly to the limit (Φ, Ψ) in $L^2 \times L^2$.

Next, observe that

$$|z_n - \Psi|_r^r \leq C \|z_n - \Psi\|_1^{1/r} \|z_n - \Psi\|^{(r-1)/r} \leq C \|z_n - \Psi\|^{(r-1)/r},$$

which implies $|z_n|_r^r \rightarrow |\Psi|_r^r$ as $n \rightarrow \infty$. Also,

$$|w_n - \Phi|_p^p \leq C \|w_n - \Phi\|_1^{1/p} \|w_n - \Phi\|^{(p-1)/p} \leq C \|w_n - \Phi\|^{(p-1)/p},$$

and hence $|w_n|_p^p \rightarrow |\Phi|_p^p$ as $n \rightarrow \infty$. The fact

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |z_n|^q |w_n|^q dx = \int_{-\infty}^{\infty} |\Psi|^q |\Phi|^q dx$$

follows by writing

$$\begin{aligned} \int_{-\infty}^{\infty} (|z_n|^q |w_n|^q - |\Psi|^q |\Phi|^q) dx &= \int_{-\infty}^{\infty} |z_n|^q (|w_n|^q - |\Phi|^q) dx \\ &\quad + \int_{-\infty}^{\infty} (|z_n|^q - |\Psi|^q) |\Phi|^q dx \end{aligned}$$

and noting that $\{(w_n, z_n)\}$ is bounded in Y . Therefore, by another application of Fatou's lemma, we get

$$(2.31) \quad \Theta(s, t) = \lim_{n \rightarrow \infty} H(w_n, z_n) \geq H(\Phi, \Psi);$$

whence $H(f, g) = \Theta(s, t)$. Thus $(\Phi, \Psi) \in \mathcal{F}_{s,t}$. Finally, since equality holds in (2.31), one has

$$\lim_{n \rightarrow \infty} (\|w_{nx}\|^2 + \|z_{nx}\|^2) = \|\Phi_x\|^2 + \|\Psi_x\|^2,$$

so $(w_n(x), z_n(x))$ converges strongly to (Φ, Ψ) in the norm of Y . \square

The following result, which we state here without proof, is a special case of Lemma I.1 of [26]. For a proof, see Lemma 2.13 of [3].

Lemma 2.10. *Suppose f_n is a bounded sequence in $H^1(\mathbb{R})$ such that, for some $R > 0$,*

$$(2.32) \quad \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} f_n^2 dx = 0.$$

Then for every $k > 2$,

$$\lim_{n \rightarrow \infty} |f_n|_k = 0.$$

We can now rule out the case of vanishing:

Lemma 2.11. *For any minimizing sequence $\{(f_n, g_n)\} \in Y$, $\gamma > 0$.*

Proof. Suppose to contrary that $\gamma = 0$. By Lemma 2.1, both $\{|f_n|\}$ and $\{|g_n|\}$ are bounded sequences in H^1 . Using Lemma (2.10), for every $k > 2$, f_n and g_n converge to 0 in L^k norm. In particular, $|f_n|_p^p \rightarrow 0$ and $|g_n|_r^r \rightarrow 0$. Since

$$\int_{-\infty}^{\infty} |f_n|^q |g_n|^q dx \leq |f_n|_{2q}^q |g_n|_{2q}^q,$$

it follows also that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_n|^q |g_n|^q dx = 0.$$

Hence

$$(2.33) \quad \Theta(s, t) = \lim_{n \rightarrow \infty} H(f_n, g_n) \geq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|f_{nx}|^2 + |g_{nx}|^2) dx \geq 0,$$

contradicting Lemma 2.1. This proves $\gamma > 0$. \square

Lemma 2.12. *There exist $s_1 \in [0, s]$ and $t_1 \in [0, t]$ such that*

$$(2.34) \quad \gamma = s_1 + t_1$$

and

$$(2.35) \quad \Theta(s_1, t_1) + \Theta(s - s_1, t - t_1) \leq \Theta(s, t).$$

Proof. Let ϵ be an arbitrary positive number. From the definition of γ , it follows that for ζ sufficiently large, we have $\gamma - \epsilon < M(\zeta) \leq M(2\zeta) \leq \gamma$. By taking ζ larger if necessary, we may also assume that $\frac{1}{\zeta} < \epsilon$. From the definition of M , we can choose N so large that, for every $n \geq N$,

$$\gamma - \epsilon < M_n(\zeta) \leq M_n(2\zeta) \leq \gamma + \epsilon.$$

Hence, for each $n \geq N$, we can find y_n such that

$$(2.36) \quad \int_{y_n - \zeta}^{y_n + \zeta} (|f_n|^2 + |g_n|^2) dx > \gamma - \epsilon \text{ and } \int_{y_n - 2\zeta}^{y_n + 2\zeta} (|f_n|^2 + |g_n|^2) dx < \gamma + \epsilon.$$

Now choose smooth functions ρ and σ on \mathbb{R} such that $\rho^2 + \sigma^2 = 1$ on \mathbb{R} , and ρ is identically 1 on $[-1, 1]$ and has support in $[-2, 2]$. Set, for $\zeta > 0$,

$$\rho_\zeta(x) = \rho(x/\zeta) \text{ and } \sigma_\zeta(x) = \sigma(x/\zeta).$$

From the definition of γ , it follows that for given $\epsilon > 0$, there exist $\zeta > 0$ and a sequence y_n such that, after passing to a subsequence, the functions defined by

$$(f_n^{(1)}(x), g_n^{(1)}(x)) = \rho_\zeta(x - y_n)(f_n(x), g_n(x))$$

and

$$(f_n^{(2)}(x), g_n^{(2)}(x)) = \sigma_\zeta(x - y_n)(f_n(x), g_n(x))$$

satisfy

$$\|f_n^{(1)}\|^2 \rightarrow s_1, \quad \|g_n^{(1)}\|^2 \rightarrow t_1, \quad \|f_n^{(2)}\|^2 \rightarrow s - s_1, \quad \text{and} \quad \|g_n^{(2)}\|^2 \rightarrow t - t_1,$$

as $n \rightarrow \infty$. Now

$$s_1 + t_1 = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|f_n^{(1)}|^2 + |g_n^{(1)}|^2) dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \rho_\zeta(|f_n|^2 + |g_n|^2) dx.$$

From (2.36), it follows that, for every $n \in \mathbb{N}$,

$$\gamma - \epsilon < \int_{-\infty}^{\infty} \rho_\zeta(|f_n|^2 + |g_n|^2) dx < \gamma + \epsilon.$$

Hence $|(s_1 + t_1) - \gamma| < \epsilon$. We claim that for all n ,

$$(2.37) \quad H(f_n^{(1)}, g_n^{(1)}) + H(f_n^{(2)}, g_n^{(2)}) \leq H(f_n, g_n) + C\epsilon$$

To see (2.37), we write

$$\begin{aligned} H(f_n^{(1)}, g_n^{(1)}) &= \int_{-\infty}^{\infty} \rho_\zeta^2 ((|f_{nx}|^2 + |g_{nx}|^2) - (a|f_n|^p + b|g_n|^r + c|f_n g_n|^q)) dx \\ &\quad + \int_{-\infty}^{\infty} (a(\rho_\zeta^2 - \rho_\zeta^p)|f_n|^p + b(\rho_\zeta^2 - \rho_\zeta^r)|g_n|^r + c(\rho_\zeta^2 - \rho_\zeta^{2q})|f_n|^q|g_n|^q) dx \\ &\quad + \int_{-\infty}^{\infty} \left((\rho'_\zeta)^2 (|f_n|^2 + |g_n|^2) + 2\rho'_\zeta \rho_\zeta (\operatorname{Re}(f_n(\bar{f}_n)_x) + \operatorname{Re}(g_n(\bar{g}_n)_x)) \right) dx. \end{aligned}$$

and observe that the last two integrals on the right hand side can be made arbitrarily uniformly small by taking ζ sufficiently large. Similarly, we can estimate for $H(f_n^{(2)}, g_n^{(2)})$. Then, (2.37) follows by adding these two estimates, because $\rho_\zeta^2 + \sigma_\zeta^2 = 1$.

Now, if $s_1, t_1, s - s_1$, and $t - t_1$ are all positive, then the claim follows by re-scaling $f_n^{(i)}$ and $g_n^{(i)}$ ($i = 1, 2$). Indeed, let

$$\alpha_n = \frac{\sqrt{s_1}}{\|f_n^{(1)}\|}, \quad \beta_n = \frac{\sqrt{t_1}}{\|g_n^{(1)}\|}, \quad \gamma_n = \frac{\sqrt{s - s_1}}{\|f_n^{(2)}\|}, \quad \theta_n = \frac{\sqrt{t - t_1}}{\|g_n^{(2)}\|},$$

which gives

$$(\alpha_n f_n^{(1)}, \beta_n g_n^{(1)}) \in \Sigma_{s_1, t_1} \quad \text{and} \quad (\gamma_n f_n^{(2)}, \theta_n g_n^{(2)}) \in \Sigma_{s - s_1, t - t_1}.$$

As all the scaling factors tend to 1 as $n \rightarrow \infty$,

$$\liminf_{n \rightarrow \infty} (H(f_n^{(1)}, g_n^{(1)}) + H(f_n^{(2)}, g_n^{(2)})) \geq \Theta(s_1, t_1) + \Theta(s - s_1, t - t_1).$$

If $s_1 = 0$ and $t_1 > 0$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} H(f_n^{(1)}, g_n^{(1)}) &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|(f_n^{(1)})_x|^2 + |(g_n^{(1)})_x|^2 - b|g_n^{(1)}|^r) dx \\ &\geq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|(g_n^{(1)})_x|^2 - b|g_n^{(1)}|^r) dx \geq \Theta(0, t_1). \end{aligned}$$

Similar estimates hold if t_1 , $s - s_1$, or $t - t_1$ are zero. Thus, in all the cases we have that the limit inferior as $n \rightarrow \infty$ of the left hand side of (2.37) $\geq \Theta(s_1, t_1) + \Theta(s - s_1, t - t_1)$. Consequently,

$$\Theta(s_1, t_1) + \Theta(s - s_1, t - t_1) \leq \Theta(s, t) + C\epsilon,$$

which proves the lemma, as ϵ is arbitrary. \square

The following lemma rules out the possibility of dichotomy of minimizing sequences:

Lemma 2.13. *For every minimizing sequence, one has $\gamma \notin (0, s + t)$.*

Proof. Suppose to the contrary that γ satisfies $0 < \gamma < s + t$. Let s_1 and t_1 be as in Lemma 2.12, and let $s_2 = s - s_1$ and $t_2 = t - t_1$. Then $s_2 + t_2 = (s + t) - \gamma > 0$, and also $s_1 + t_1 = \gamma > 0$. Furthermore, $s_1 + s_2 = s > 0$ and $t_1 + t_2 = t > 0$. Therefore Lemma 2.8 implies that (2.23) holds. But this contradicts (2.35) and thus, lemma follows. \square

Thus all the preliminaries for the proofs of Theorem 1.1 have been established. We are now able to prove statements (a)-(d) of Theorem 1.1.

Proof of Theorem 1.1. From Lemmas 2.11 and 2.13, it follows that every minimizing sequence must be compact, i.e., $\gamma = s + t$. Then the statement (a) of the Theorem 1.1 follows from Lemma 2.9.

To see the validity of statement (b), notice that (Φ, Ψ) is in the minimizing set $\mathcal{F}_{s,t}$ for $\Theta(s, t)$, and so minimizes $H(u, v)$ subject to $Q(u)$ and $Q(v)$ being held constant, the Lagrange multiplier principle asserts that there exist real numbers ω_1 and ω_2 such that

$$(2.38) \quad \delta H(\Phi, \Psi) + \omega_1 \delta Q(\Phi) + \omega_2 \delta Q(\Psi) = 0,$$

where δ denotes the Fréchet derivative. Computing the associated Fréchet derivatives we see that the equations

$$(2.39) \quad \begin{cases} -\Phi'' + \omega_1 \Phi = \alpha |\Phi|^{p-2} \Phi + \tau |\Psi|^q |\Phi|^{q-2} \Phi, \\ -\Psi'' + \omega_2 \Psi = \beta |\Psi|^{r-2} \Psi + \tau |\Phi|^q |\Psi|^{q-2} \Psi, \end{cases}$$

hold, at least in the sense of distributions. A straightforward bootstrapping argument (cf. Lemma 1.3 of [38]) shows that distributional solutions are also classical solutions.

Multiplying the first equation in (2.39) by $\bar{\Phi}$ and the second equation by $\bar{\Psi}$, and integrating over \mathbb{R} , we obtain

$$(2.40) \quad \begin{aligned} \int_{-\infty}^{\infty} (|\Phi'|^2 - \alpha|\Phi|^p - \tau|\Phi|^q|\Psi|^q) \, dx &= -\omega_1 \int_{-\infty}^{\infty} |\Phi|^2 \, dx = -\omega_1 s, \\ \int_{-\infty}^{\infty} (|\Psi'|^2 - \beta|\Psi|^r - \tau|\Phi|^q|\Psi|^q) \, dx &= -\omega_2 \int_{-\infty}^{\infty} |\Psi|^2 \, dx = -\omega_2 t. \end{aligned}$$

From Lemma 2.4, applied to $(f_n, g_n) = (\Phi, \Psi)$, we have that $\omega_1, \omega_2 > 0$. This proves assertion (b) of Theorem 1.1.

We now prove statement (c) of Theorem 1.1. We write

$$\Phi(x) = e^{i\theta_1(x)} |\Phi(x)| \quad \text{and} \quad \Psi(x) = e^{i\theta_2(x)} |\Psi(x)|,$$

where $\theta_1, \theta_2 : \mathbb{R} \rightarrow \mathbb{R}$. Define $\tilde{\phi}(x) = |\Phi(x)|$ and $\tilde{\psi}(x) = |\Psi(x)|$. Note that $(\tilde{\phi}, \tilde{\psi})$ is also in $\mathcal{F}_{s,t}$, as follows from Lemma 2.5. Therefore, $(\tilde{\phi}, \tilde{\psi})$ satisfies the Lagrange multiplier equations

$$(2.41) \quad \begin{cases} -\tilde{\phi}'' + \omega_1 \tilde{\phi} = \alpha|\tilde{\phi}|^{p-2}\tilde{\phi} + \tau|\tilde{\psi}|^q|\tilde{\phi}|^{q-2}\tilde{\phi}, \\ -\tilde{\psi}'' + \omega_2 \tilde{\psi} = \beta|\tilde{\psi}|^{r-2}\tilde{\psi} + \tau|\tilde{\phi}|^q|\tilde{\psi}|^{q-2}\tilde{\psi}, \end{cases}$$

(The Lagrange multipliers are determined by the equation (2.40) and this equation stay same when (Φ, Ψ) is replaced by $(\tilde{\phi}, \tilde{\psi})$, and hence the Lagrange multipliers are unchanged.) We compute

$$(2.42) \quad \Phi'' = e^{i\theta_1} \left(\omega_1 \tilde{\phi} - \alpha|\tilde{\phi}|^{p-2}\tilde{\phi} - \tau|\tilde{\psi}|^q|\tilde{\phi}|^{q-2}\tilde{\phi} - (\theta_1')^2 \tilde{\phi} + 2i\theta_1' \tilde{\phi}' + i\theta_1'' \tilde{\phi} \right).$$

On the other hand, from the first equation of (2.39), we have that

$$(2.43) \quad \Phi'' = e^{i\theta_1} \left(\omega_1 \tilde{\phi} - \alpha|\tilde{\phi}|^{p-2}\tilde{\phi} - \tau|\tilde{\psi}|^q|\tilde{\phi}|^{q-2}\tilde{\phi} \right).$$

From (2.42) and (2.43), we obtain

$$(\theta_1'(x))^2 \tilde{\phi}(x) - 2i\theta_1'(x) \tilde{\phi}'(x) - i\theta_1''(x) \tilde{\phi}(x) = 0.$$

Equating the real part of the last equation, we conclude that $\theta_1'(x) = 0$, and hence $\theta_1(x)$ is constant. Similarly, $\theta_2(x)$ is constant.

Next, for any $\xi > 0$, define the function $K_\xi(x)$ by

$$K_\xi(x) = \frac{1}{2\sqrt{\xi}} e^{-\sqrt{\xi}|x|}.$$

A calculation using Fourier transform shows that the operators $\omega_1 - \partial_{xx}$ and $\omega_2 - \partial_{xx}$ appearing in (2.41) are invertible on H^1 , with inverse given by convolution with the functions K_{ω_1} and K_{ω_2} respectively. Then, the Lagrange multiplier equations associated with $(\tilde{\phi}, \tilde{\psi})$ can be written as

$$\tilde{\phi} = K_{\omega_1} \star \left(\alpha|\tilde{\phi}|^{p-2}\tilde{\phi} + \tau|\tilde{\psi}|^q|\tilde{\phi}|^{q-2}\tilde{\phi} \right), \quad \tilde{\psi} = K_{\omega_2} \star \left(\beta|\tilde{\psi}|^{r-2}\tilde{\psi} + \tau|\tilde{\phi}|^q|\tilde{\psi}|^{q-2}\tilde{\psi} \right)$$

Since the convolutions of K_{ω_1} and K_{ω_2} with functions that are everywhere non-negative and not identically zero must produce everywhere positive functions, it follows that $\tilde{\phi}(x) > 0$ and $\tilde{\psi}(x) > 0$ for all $x \in \mathbb{R}$. This completes proof of statement Theorem 1.1(c).

It remains to prove part (d) of Theorem 1.1. Suppose to the contrary that $\mathcal{F}_{s,t}$ is unstable. Then there exist a number $\epsilon > 0$, a sequence of times t_n , and a sequence $(u_n(x, 0), v_n(x, 0))$ in Y such that for all n ,

$$(2.44) \quad \inf\{\|(u_n(x, 0), v_n(x, 0)) - (\Phi, \Psi)\|_Y : (\Phi, \Psi) \in \mathcal{F}_{s,t}\} < \frac{1}{n};$$

and

$$(2.45) \quad \inf\{\|(u_n(\cdot, t_n), v_n(\cdot, t_n)) - (\Phi, \Psi)\|_Y : (\Phi, \Psi) \in \mathcal{F}_{s,t}\} \geq \epsilon,$$

where $(u_n(x, t), v_n(x, t))$ solves (1.2) with initial data $(u_n(x, 0), v_n(x, 0))$. From (2.44) and the continuity of the functionals H and Q , we have

$$(2.46) \quad \begin{aligned} \lim_{n \rightarrow \infty} H(u_n(x, 0), v_n(x, 0)) &= \Theta(s, t), \\ \lim_{n \rightarrow \infty} Q(u_n(x, 0)) &= s, \\ \lim_{n \rightarrow \infty} Q(v_n(x, 0)) &= t. \end{aligned}$$

Denote $R_n = u_n(\cdot, t_n)$ and $S_n = v_n(\cdot, t_n)$. Since $H(u, v)$ and $Q(u)$ are conserved quantities, then (2.46) implies

$$\begin{aligned} \lim_{n \rightarrow \infty} H(R_n, S_n) &= \Theta(s, t), \\ \lim_{n \rightarrow \infty} Q(R_n) &= s, \\ \lim_{n \rightarrow \infty} Q(S_n) &= t. \end{aligned}$$

Therefore $\{(R_n, S_n)\}$ is a minimizing sequence for $\Theta(s, t)$. Now, by the first part of Theorem 1.1, there exists a subsequence $\{(R_{n_k}, S_{n_k})\}$, $\{y_k\} \subset \mathbb{R}$, and a pair $(\Phi, \Psi) \in \mathcal{F}_{s,t}$ such that

$$(2.47) \quad \lim_{k \rightarrow \infty} \|(R_{n_k}(\cdot + y_k), S_{n_k}(\cdot + y_k)) - (\Phi, \Psi)\|_Y = 0.$$

Then, for some sufficiently large k ,

$$\|(R_{n_k}(\cdot + y_k), S_{n_k}(\cdot + y_k)) - (\Phi, \Psi)\|_Y < \epsilon,$$

and hence

$$(2.48) \quad \|(R_{n_k}, S_{n_k}) - (\Phi(\cdot - y_k), \Psi(\cdot - y_k))\|_Y < \epsilon.$$

Since $\mathcal{F}_{s,t}$ is invariant under translations, $(\Phi(\cdot - y_k), \Psi(\cdot - y_k))$ belongs to $\mathcal{F}_{s,t}$, contradicting (2.45), and hence the minimizing set $\mathcal{F}_{s,t}$ must be stable. \square

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